Tasks 04 - Differentiation

Mathematics for Master Students

These problems accompany Session 04 of the tutorial. Work through them at your own pace before the next session. We start Session 05 with your questions. On the website, every problem has a fully worked solution you can expand to check yourself. The PDF version contains only the problems, so you can print it and use it as a worksheet.

The problems are staged: the first two are warm-ups, the middle ones practise the core techniques, and the last two are at the level of the mock exam. Throughout, name the structure first (sum, product, quotient or chain) and the right rule follows automatically.

Problem 1: Power Rule Warm-Up

Differentiate each function. For (c) and (d), **rewrite as a power first**, differentiate, then translate back.

a) $f(x) = x^6$
b) $g(x) = 4x^3 - 5x^2 + 2x - 11$
c) $h(x) = \frac{3}{x}$
d) $k(x) = \sqrt{x}$

Problem 2: Product, Quotient and Chain

Differentiate each function using the named rule.

a) $u(x) = (x^2 + 3)(2x - 1)$: product rule (then check by expanding first)
b) $v(x) = \frac{x-1}{x+2}$: quotient rule
c) $w(x) = (3x + 1)^5$: chain rule

Problem 3: Exponentials and Logarithms

Differentiate. Remember $(e^x)' = e^x$ and $(\ln x)' = \frac{1}{x}$, combined with the chain rule where needed.

a) $f(x) = e^{4x}$
b) $g(x) = \ln(x^2 + 1)$
c) $h(x) = x^2 e^x$
d) $m(x) = 5 \ln x - 2e^x$

Problem 4: Tangent Lines

The tangent line to f at $x = a$ has slope $f'(a)$ and passes through $(a, f(a))$, so its equation is $y = f(a) + f'(a)(x - a)$. Consider $f(x) = x^2 - 6x + 10$.

a) Find $f'(x)$, and the slope of the tangent at $x = 1$ and at $x = 4$.
b) Find the equation of the tangent line at $x = 4$, simplified to the form $y = mx + b$.

c) At which x is the tangent horizontal? What is special about that point?

Problem 5: Marginal Cost

A workshop's monthly cost of producing q units is

$$C(q) = 0.1q^2 + 15q + 5000 \quad (\text{in } €) \ .$$

a) Find the marginal cost function $C'(q)$.
b) Compute $C'(100)$ and state its **units**. Interpret the number in one sentence.
c) Compute the *exact* cost of the 101st unit, $C(101) - C(100)$, and compare it with your answer from (b).
d) Is marginal cost increasing or decreasing? Justify with $C''(q)$.

Problem 6: Elasticity of Demand

Weekly demand for a courier's express service is $D(p) = 400 - 8p$, where p is the price in €. Recall the elasticity $\varepsilon = \frac{p}{D(p)} \cdot D'(p)$.

a) Compute $D'(p)$, then the elasticity at $p = 20$. Classify it as elastic or inelastic, and say what a small price increase does to revenue.
b) Repeat at $p = 35$.
c) Find the price at which $\varepsilon = -1$ (unit elastic). Confirm that revenue $R(p) = p \cdot D(p)$ peaks there.

Problem 7: Name the Structure

These are exam-level: each needs *two* rules. Name the outer structure first, then differentiate.

a) $f(x) = x^3 e^{2x}$
b) $g(x) = \sqrt{x^2 + 4}$
c) $h(x) = \frac{\ln x}{x^2}$

Problem 8: Growth and Curvature

A warehouse's stored volume grows continuously as $V(t) = 200\, e^{0.1t}$ (in m^3 after t weeks).

a) Find $V'(t)$ and $V''(t)$.
b) Show that $V'(t) = 0.1 \cdot V(t)$, and say what this means in words.
c) Compute the growth rate at $t = 0$ and at $t = 10$ (you may leave the second answer as a multiple of e, then give a decimal).
d) Is V convex or concave? What does that say about how the volume grows?