Tasks 03 - Functions

Mathematics for Master Students

These problems accompany Session 03 of the tutorial. Work through them at your own pace before the next session. We start Session 04 with your questions. On the website, every problem has a fully worked solution you can expand to check yourself. The PDF version contains only the problems, so you can print it and use it as a worksheet.

The problems are staged: the first two are warm-ups, the middle ones practise the core techniques, and the last two are at the level of the mock exam.

Problem 1: Evaluating a Function

Let $f(x) = x^2 + 4x$.

a) Compute $f(3)$, $f(0)$ and $f(-2)$.
b) Compute $f(\frac{1}{2})$.
c) Compute $f(a+h)$ and expand the result.
d) Check with $a = h = 1$ that $f(a+h) \neq f(a) + f(h)$.

Problem 2: Domain, Codomain, Range

A parcel service charges by weight class. For the classes $A = \{1, 2, 3, 4, 5\}$ (kilograms, rounded up), the fee in € is given by

$$f : A \to \mathbb{R}, \quad f(x) = 4x + 6$$

a) State the domain and the codomain of f, and compute its range.
b) Now consider $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 + 2$. What is the range of g, and why is it not equal to the codomain?
c) The service also keeps a data table assigning to each *customer* the order values of all their orders. One customer placed three orders. Is "customer $\mapsto$ order value" a function? What about "order $\mapsto$ order value"?
d) True or false: the range of a function is always a subset of its codomain.

Problem 3: Maximal Domains

If no domain is stated, we use the maximal domain: the largest set of real numbers on which the formula is defined. Find it for each function and write it in interval notation.

a) $f(x) = \frac{7}{2x-10}$
b) $g(x) = \sqrt{9 - 3x}$
c) $h(x) = \ln(2x + 8)$
d) $k(x) = \frac{\sqrt{x+1}}{x-2}$

Problem 4: Break-Even Analysis

A manufacturer of e-bike batteries has fixed costs of €30,000 per month. Each battery costs €75 to produce and sells for €125. Let x be the number of batteries produced and sold per month.

a) Write down the monthly cost function $C(x)$ and the revenue function $R(x)$. What do the slope and the intercept of C mean in business terms?
b) Compute cost, revenue and the resulting profit or loss at $x = 400$ units.
c) Find the **break-even point**: the quantity at which revenue exactly covers cost. Verify your answer.
d) Write down the profit function $P(x) = R(x) - C(x)$. How many batteries must be sold to reach a monthly profit of €20,000?

Problem 5: Maximal Revenue and the Vertex

A regional carrier estimates the weekly demand for a new express connection as $q(p) = 600 - 15p$, where p is the price in € and q the number of booked shipments.

a) Write down the revenue function $R(p) = p \cdot q(p)$. What type of function is it, and which way does its parabola open?
b) Find the revenue-maximal price with the vertex formula $p = -\frac{b}{2a}$, and compute the maximal revenue.
c) Confirm your result by **completing the square**: bring $R(p)$ into the vertex form $R(p) = a(p - p_0)^2 + R_0$ and read off the vertex. Why does this form *prove* that the vertex is a maximum?
d) For which prices is the model meaningful at all? Give the interval and interpret its endpoints.

Problem 6: Composition and Inverses

An online shop applies a 20% discount, $d(p) = 0.8p$, and adds a flat €5 shipping fee, $s(p) = p + 5$, to a basket value p.

a) Compute both compositions $(s \circ d)(p)$ and $(d \circ s)(p)$. Which order is cheaper for the customer, and by how much?
b) Evaluate both compositions for a basket of $p = 60$.
c) The shop's demand forecast is $D(p) = 800 - 25p$. Find the inverse function D^{-1} with the swap-and-solve recipe, interpret it, and verify it with the pair $p = 20$.
d) Why does $h(x) = x^2$ on $\mathbb{R}$ have no inverse, and how can the domain be repaired so that it does?

Problem 7: Exponential Depreciation

A logistics company buys a forklift for €80,000. Its resale value depreciates by 15% per year, so after t years it is worth

$$V(t) = 80,000 \cdot 0.85^t$$

a) Explain the two constants in the formula. Then show, using an exponent rule, that $V(t+1) = 0.85 \cdot V(t)$ for every t: each year costs the same *percentage*, not the same amount.
b) Compute the value after 3 years and after 5 years.
c) The company plans to replace the forklift once its value drops below €20,000. After how many years does that happen? Solve exactly with logarithms, then give the first full year in which the value is below the threshold.
d) Find the "half-value time": how long until the forklift is worth half its purchase price? Use your result to double-check c).

Problem 8: Reading Function Behaviour

A planning team works with five model functions:

$$T(x) = 40\sqrt{x} \quad (x \geq 0) \qquad u(x) = \frac{600}{x} + 8 \quad (x > 0) \qquad V(t) = 20,000 \cdot 0.9^t \quad (t \geq 0)$$

$$L(x) = 120 - 4x \qquad q(x) = (x-6)^2 + 2$$

(T: daily throughput for x workers; u: cost per unit at batch size x; V: resale value; L: stock level after x weeks; q: unit cost at machine speed x.)

a) Classify each function as (strictly) increasing, (strictly) decreasing, or neither, with a one-line reason; no derivatives needed.
b) Classify each function as convex, concave, or both, using the chord test from the definition (chord above the graph = convex, below = concave).
c) Which of the five functions are one-to-one on the given domain, and therefore invertible?
d) Give a lower bound for u, V and q. Is T bounded above?