Tasks 02 - Sets

Mathematics for Master Students

These problems accompany Session 02 of the tutorial. Work through them at your own pace before the next session. We start Session 03 with your questions. On the website, every problem has a fully worked solution you can expand to check yourself. The PDF version contains only the problems, so you can print it and use it as a worksheet.

The problems are staged: the first two are warm-ups, the middle ones practise the core techniques, and the last two are at the level of the mock exam.

Problem 1: Element or Subset?

A freight forwarder offers containers with the capacities (in tons) $D = \{2, 4, 8, 16\}$. Decide for each statement whether it is **true** or **false**, and justify your answer in one sentence. The key question is always: does the symbol relate an *element* to a set, or *two sets*?

a) $4 \in D$
b) $\{4\} \in D$
c) $\{2, 16\} \subseteq D$
d) $6 \notin D$
e) $\varnothing \subseteq D$
f) $D \subset D$

Problem 2: Set-Builder and Listing

Translate between the two ways of describing a set. Remember this tutorial's convention: $\mathbb{N} = \{1, 2, 3, \ldots\}$ does **not** contain 0.

a) List the elements of $\{x \in \mathbb{N} : x \text{ is odd and } x < 10\}$.
b) List the elements of $\{x \in \mathbb{Z} : -2 \leq x < 3\}$.
c) Write $\{5, 10, 15, 20, 25\}$ in set-builder notation.
d) List the elements of $\{x \in \mathbb{N} : x^2 < 20\}$.

Problem 3: Set Operations and De Morgan's Laws

Work inside the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let

$$A = \{1, 3, 5, 7, 9\} \quad (\text{the odd numbers}) \qquad B = \{6, 7, 8, 9, 10\} \quad (\text{the numbers above } 5)$$

a) Compute $A \cup B$ and $A \cap B$.
b) Compute $A \setminus B$ and $B \setminus A$. Are they equal?
c) Compute the complements A^c and B^c.
d) Verify **both** De Morgan's laws for these sets: compute $(A \cup B)^c$ and $A^c \cap B^c$, then $(A \cap B)^c$ and $A^c \cup B^c$.

Problem 4: Counting with Inclusion-Exclusion

A retailer audits its 120 suppliers. The audit finds that 65 suppliers hold a quality certification, 48 hold an environmental certification, and 27 hold **both**.

a) How many suppliers hold **at least one** of the two certifications?
b) How many suppliers hold **no** certification at all?
c) How many suppliers hold the quality certification **only** (and not the environmental one)?
d) A colleague simply adds $65 + 48 = 113$ and reports "113 certified suppliers". Explain in one sentence what went wrong.

Problem 5: Cartesian Products and Counting Routes

A carrier serves the hubs $H = \{\text{Hamburg}, \text{Rotterdam}, \text{Antwerp}\}$ with departures on the days $D = \{\text{Mon}, \text{Thu}\}$. Every combination of hub and day is one **departure slot**, an ordered pair (hub, day).

a) Write down the set $H \times D$ by listing all its elements.
b) Compute $|H \times D|$ using the counting rule. Does it match your list?
c) Is $(\text{Mon}, \text{Hamburg}) \in H \times D$? Justify your answer.
d) The carrier expands to 4 hubs and 5 departure days. How many departure slots are there now? And conversely: a competitor's schedule contains 24 slots using 4 hubs: on how many days does it operate?

Problem 6: Grouping Matters, Three Sets

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$ and $C = \{4, 5, 6\}$. A colleague claims: "$A \cap B \cup C$ is unambiguous: parentheses are just decoration." Test that claim.

a) Compute $A \cap (B \cup C)$: first the union, then the intersection.
b) Compute $(A \cap B) \cup C$: first the intersection, then the union.
c) Compare the two results. What does this tell you about the colleague's claim?
d) Now compute $(A \cap B) \cup (A \cap C)$ and compare with a). Which general law does this illustrate?

Problem 7: Complements Reverse Inclusion

An online shop tracks its customer base $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (customer IDs). The **express-delivery** customers are $A = \{2, 3\}$ and the **premium** customers are $B = \{2, 3, 5, 7\}$: every express customer is premium, i.e. $A \subseteq B$.

a) Compute A^c and B^c. Then check both directions: is $B^c \subseteq A^c$? Is $A^c \subseteq B^c$?
b) Explain why $A \subseteq B \Rightarrow B^c \subseteq A^c$ holds for **all** sets A, B, not just this example. (Hint: take an arbitrary $x \in B^c$ and argue that $x \in A^c$.)
c) A colleague claims the direction is preserved instead: "$A \subseteq B \Rightarrow A^c \subseteq B^c$." Disprove this with a concrete counterexample.

Problem 8: Power Sets and Product Bundles

A category manager can bundle any selection of the products $S = \{P, Q, R\}$ into a promotion.

a) List **all** elements of the power set $\mathcal{P}(S)$ and confirm that their number matches the formula $|\mathcal{P}(S)| = 2^{|S|}$.
b) How many of these subsets contain the flagship product P? List them. Can you find the count *without* listing?
c) How many bundles contain **at least one** product?
d) Next quarter the assortment grows to 5 products, and marketing requires every promotion to bundle **at least two** of them. How many admissible bundles are there?