Session 05 - Single Variable Optimization

Mathematics for Master Students

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Recap & Your Questions

Session 04 in a Nutshell

- The derivative $f'(a)$ is the slope of the **tangent**, the instantaneous rate of change at a
- Power, sum, product, quotient and chain rule differentiate every gallery function
- e^x is its own derivative; $\ln x$ has derivative $\frac{1}{x}$
- f'' measures curvature: $f'' > 0$ convex, $f'' < 0$ concave, the chord intuition, formalised
- Marginal cost $C'(q) \approx$ cost of one more unit; where $R'(q) = C'(q)$ hides the best output
- Today: we turn a **flat tangent** into a recipe for finding the best decision

Common Issues in the Homework

- **The inner derivative went missing:** with $V(t) = 200e^{0.1t}$, the chain rule gives $V'(t) = 20e^{0.1t}$, the factor 0.1 is not optional, so the answer is not $200e^{0.1t}$
- **Quotient order flipped:** in $\left(\frac{\ln x}{x^2}\right)'$ the numerator is $f'g - fg'$, the minus acts on the second product; swapping the terms flips the sign
- **A rate read as a total:** $C'(100) = 35$ means the **101st unit** costs about €35, *not* that 100 units cost €35 (that total is $C(100) = 7500$, i.e. €7500)

...

Tip

Which tasks gave you trouble? Bring them up **now**. We take the time to go through them before we move on to today's topic.

Warm-Up: Find the Flat Spot

Question

Differentiate $f(x) = 2x^2 - 8x + 3$. At which x is the tangent **horizontal**?

...

- Differentiate term by term: $f'(x) = 4x - 8$

- Horizontal tangent means slope zero: $4x - 8 = 0 \Rightarrow x = 2$
- At $x = 2$ the parabola turns, its lowest point, the vertex
- That flat spot is exactly where today's hunt for optima begins

Today's Plan

- **Stationary points**, a flat tangent flags a candidate: solve $f'(x) = 0$
- **The second-order condition**: does f'' say peak or valley?
- **Local vs global**: closed intervals and business boundaries
- **Optimisation in business**: profit maximisation and the economic order quantity

Stationary Points & the First-Order Condition

What Does "Optimal" Mean?

Optimisation means finding the input x that makes $f(x)$ as large or as small as possible.

- **Local maximum** at x^*: $f(x^*) \geq f(x)$ for all nearby x, a peak
- **Local minimum** at x^*: $f(x^*) \leq f(x)$ for all nearby x, a valley
- **Global** (absolute) extremum: the largest or smallest value on the whole domain
- Business speaks this language daily: *most* profit, *least* cost, *best* order size

Where Can an Optimum Hide?

An extremum of a smooth function can only sit in one of three kinds of place:

- A **stationary point**, where $f'(x) = 0$ (the tangent is flat)
- A **boundary** of the domain, the edge of what is allowed
- A **kink**, where f' does not exist (Session 4's non-differentiable points)

. . .

For the smooth business functions in this tutorial, the first two cover almost everything, and stationary points come first.

Your Turn: Where Does It Hide?

ℹ Question

$f(x) = |x - 3|$ has its minimum at $x = 3$. Which kind of place is that?

(a) A stationary point **(b)** A boundary of the domain

(c) A kink **(d)** It has no minimum

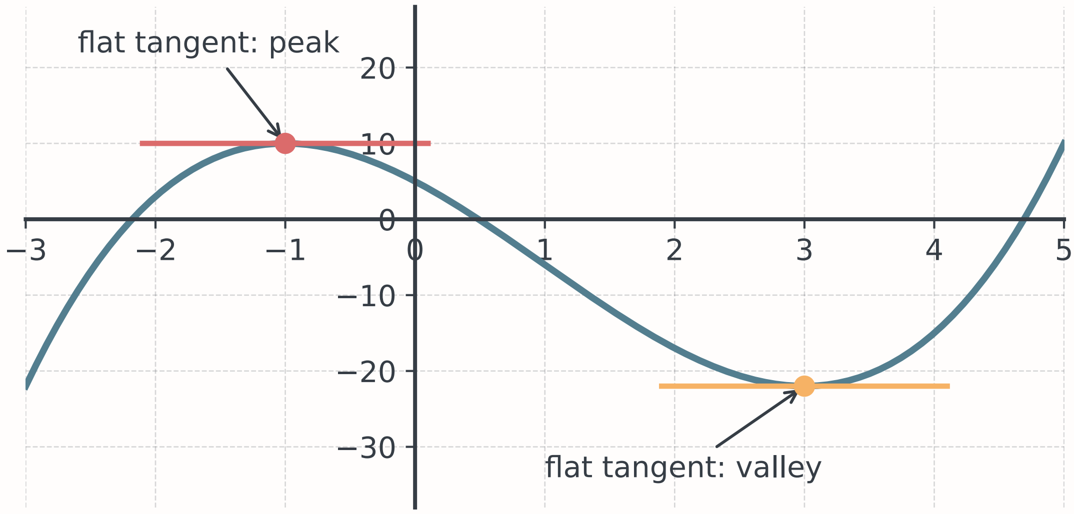
. . .

(c): the graph is V-shaped, and at the tip no tangent exists, so $f'(3)$ is undefined and the FOC cannot find it.

. . .

Session 4's non-differentiable points are not exotic: tiered tariffs and absolute deviations (forecast errors!) produce kinks all the time.

The Flat-Tangent Idea



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At a smooth peak or valley the graph momentarily stops rising or falling, the tangent is horizontal, so its slope $f'(x)$ is **zero**.

The First-Order Condition

A point x^* with $f'(x^*) = 0$ is called a stationary point (or critical point).

$$f'(x^*) = 0 \qquad \text{the first-order condition (FOC)}$$

- It is a necessary condition: every smooth interior extremum **must** be stationary
- So the FOC gives us the **candidates**, solve $f'(x) = 0$ to find them all
- It is not sufficient: a flat tangent alone does not prove a peak or a valley
- Second step (next section) sorts the candidates into maxima, minima and neither

Finding the Candidates

Take $f(x) = x^3 - 3x^2 - 9x + 5$. Where are its stationary points?

- **Differentiate:** $f'(x) = 3x^2 - 6x - 9$
- **Factor:** $f'(x) = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$
- **Set to zero:** $3(x - 3)(x + 1) = 0 \Rightarrow x = -1 \ \text{ or } \ x = 3$
- Two candidates: $x = -1$ and $x = 3$, but which is a peak, which a valley?

...

Tip

Always **factor** $f'(x)$ if you can, the stationary points are just its roots, and a factored form hands them to you directly.

Your Turn: Find the Stationary Points

Question

Where are the stationary points of $f(x) = x^3 - 12x$? Shout all of them.

...

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2) = 0 \quad \Rightarrow \quad x = 2 \text{ or } x = -2$$

...

Common Mistake

Solving $x^2 = 4$ as "$x = 2$" loses half the candidates. A square root has two signs; keep them both until the second-order test sorts them.

Marginal Revenue Meets Marginal Cost

Session 4 promised: the best output is where marginal revenue meets marginal cost. Here is why.

Profit is revenue minus cost: $\pi(q) = R(q) - C(q)$.

...

The first-order condition $\pi'(q) = 0$ becomes:

$$\pi'(q) = R'(q) - C'(q) = 0 \quad \Leftrightarrow \quad R'(q) = C'(q)$$

...

- In words: keep producing while the next unit earns more than it costs
- Stop where the extra revenue exactly equals the extra cost, $R' = C'$
- The abstract "$f'(x) = 0$" **is** the business rule "marginal revenue = marginal cost"

Your Turn: Where Do They Meet?

Question

Marginal revenue is $R'(q) = 80 - 2q$, marginal cost is $C'(q) = 20 + q$. At which output is profit maximal? Shout the number.

...

$$80 - 2q = 20 + q \quad \Rightarrow \quad 60 = 3q \quad \Rightarrow \quad q = 20$$

. . .

Below 20 the next unit earns more than it costs, above 20 it costs more than it earns: the gap between revenue and cost is widest at $q = 20$.

Your Turn: Is a Flat Tangent Enough?

> **Question**
>
> A function satisfies $f'(2) = 0$. Does that alone tell you $x = 2$ is a maximum or a minimum?

. . .

- **No**, $f'(2) = 0$ only makes $x = 2$ a candidate
- The tangent is flat at peaks, at valleys, **and** at saddle points like $f(x) = x^3$ at 0
- The FOC narrows the search; deciding the *type* needs a second test, coming up next

Classifying Candidates: the Second-Order Condition

Peak or Valley? The Second Derivative Decides

At a stationary point x^*, curvature settles the question, Session 4's f'' returns:

- $f''(x^*) > 0$: graph is convex (bends upward), the valley of a bowl → **local minimum**
- $f''(x^*) < 0$: graph is concave (bends downward), the top of a hill → **local maximum**
- $f''(x^*) = 0$: the test is inconclusive, could be either, or neither
- The chord picture, made into a test: convex holds water, concave spills it

The Recipe in Four Steps

Every optimisation problem in this tutorial runs through the same four steps:

1. **Differentiate:** compute $f'(x)$
2. **Solve** $f'(x) = 0$: the stationary points are the candidates (FOC)
3. **Differentiate again** and evaluate f'' at each candidate
4. **Classify:** $f'' > 0$ minimum, $f'' < 0$ maximum, $f'' = 0$ undecided (SOC)

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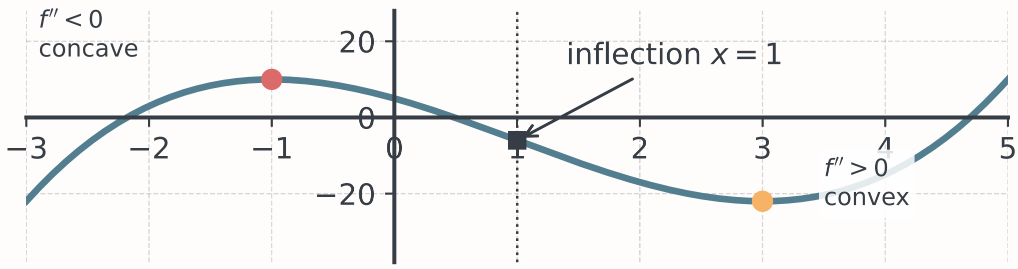
> **Tip**
>
> Say the goal out loud first, *maximise* or *minimise*, so you know which sign of f'' you are hoping for.

Classifying Our Candidates



- $f''(x) = 6x - 6$. At $x = -1$: $f''(-1) = -12 < 0$ → local maximum, value $f(-1) = 10$
- At $x = 3$: $f''(3) = 12 > 0$ → local minimum, value $f(3) = -22$
- Where $f'' = 0$, at $x = 1$, the bend switches: an **inflection point** $(1, -6)$

Your Turn: Classify the Candidates

i Question

Back to $f(x) = x^3 - 12x$ with stationary points $x = 2$ and $x = -2$. Which is which?

(a) Maximum at 2, minimum at -2

(b) Minimum at 2, maximum at -2

(c) Both are minima

(d) The test is inconclusive

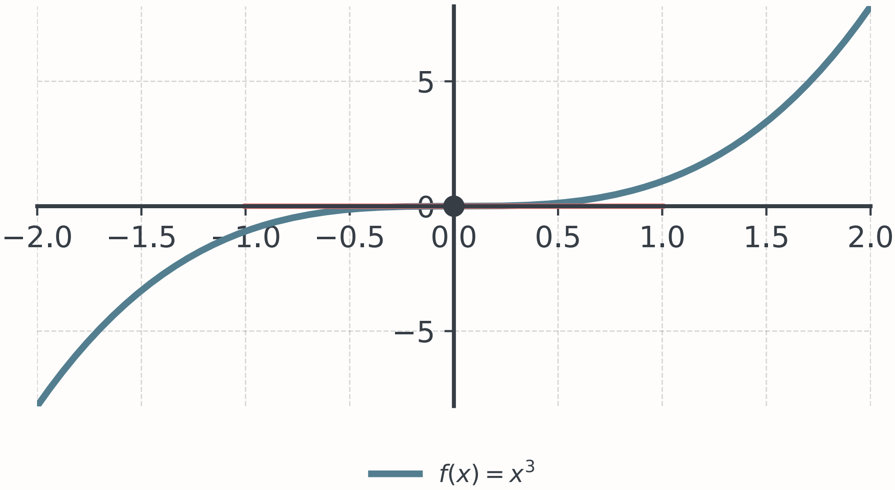
...

(b): $f''(x) = 6x$, so $f''(2) = 12 > 0$ (valley) and $f''(-2) = -12 < 0$ (peak).

...

Values: $f(2) = 8 - 24 = -16$ and $f(-2) = -8 + 24 = 16$, a local minimum of -16 and a local maximum of 16.

When the Test Is Silent: $f'' = 0$



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For $f(x) = x^3$: $f'(0) = 0$ **and** $f''(0) = 0$. The tangent is flat, yet the graph keeps rising, a saddle point, neither peak nor valley.

The FOC Is Never the Whole Story

> ⚠ Common Mistake
>
> Declaring "$f'(x^*) = 0$, therefore it is a maximum". A stationary point is only a **candidate**. Always check the second-order condition (or watch how f' changes sign across x^*) before naming it a max or a min.

...

- **Alternative, the first-derivative test:** if f' goes $+ \to -$ at x^*, it is a maximum; $- \to +$, a minimum
- If f' keeps its sign across x^* (as for x^3), it is neither
- The first-derivative test always works, handy exactly when $f'' = 0$ falls silent

The First-Derivative Test as a Table

For $f(x) = x^3 - 3x^2 - 9x + 5$, $f'(x) = 3(x-3)(x+1)$. One test point per interval:

Interval	$x < -1$	$-1 < x < 3$	$x > 3$
test point	$x = -2$	$x = 0$	$x = 4$
$f'(x)$	$3 \cdot (-5) \cdot (-1) = 15$	$3 \cdot (-3) \cdot 1 = -9$	$3 \cdot 1 \cdot 5 = 15$
sign	$+$	$-$	$+$

. . .

- At $x = -1$ the sign flips $+ \to -$: rising, then falling, a maximum
- At $x = 3$ it flips $- \to +$: falling, then rising, a minimum
- Same verdict as f'', and it also works when $f'' = 0$

Your Turn: Max or Min?

> **Question**
>
> At a stationary point x^* of a cost function, you compute $f''(x^*) = 8 > 0$. Is x^* a local maximum or a local minimum?

. . .

- $f''(x^*) > 0$ means the graph is convex there, a valley
- So x^* is a local minimum, exactly what you want when minimising cost
- Mnemonic: **positive** second derivative → holds water → **minimum**

Local vs Global & Boundaries

Local Is Not Global

A local optimum is best only in its neighbourhood, a bigger peak may sit elsewhere.

- The cubic's local min at $(3, -22)$ is a valley, but f dives far lower as $x \to -\infty$
- On an unbounded domain, a global optimum may not exist at all
- **Extreme Value Theorem:** a continuous function on a *closed* interval $[a, b]$ always attains both a global max and a global min
- Business is usually bounded (quantities live in $[0, \text{capacity}]$) so globals exist

The Closed-Interval Recipe

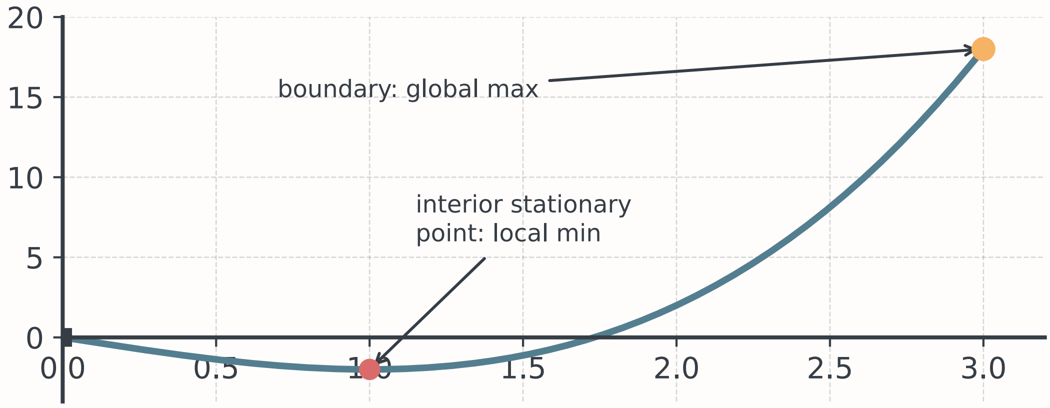
To find the global max and min of f on $[a, b]$, compare a short list:

1. Find every stationary point in (a, b): solve $f'(x) = 0$
2. Evaluate f at each of them
3. Evaluate f at both **endpoints**, $f(a)$ and $f(b)$
4. The largest value is the global max; the smallest is the global min

. . .

> **Common Mistake**
>
> Finding stationary points but forgetting the endpoints. The global optimum often sits at a boundary, skip it and you get the wrong answer.

A Boundary Can Win



Comparing the Candidates

Maximise $g(x) = x^3 - 3x$ on $[0, 3]$:

x	0 (endpoint)	1 (stationary)	3 (endpoint)
$g(x)$	0	-2	$\mathbf{18}$

...

The only interior stationary point is a **minimum**; the global max 18 sits at the boundary $x = 3$.

Your Turn: Global on an Interval

i Question

Find the global maximum of $f(x) = x^2 - 4x$ on $[0, 5]$. Shout the maximum **value**.

...

- Stationary point: $f'(x) = 2x - 4 = 0 \Rightarrow x = 2$, with $f(2) = 4 - 8 = -4$
- Endpoints: $f(0) = 0$ and $f(5) = 25 - 20 = 5$
- Compare $\{-4, 0, 5\}$: the global maximum is 5, at the endpoint $x = 5$; the stationary point is the global *minimum*

...

⚠ Common Mistake

Reporting -4 because “that is where $f' = 0$”. An upward parabola has no interior maximum at all: on a closed interval its maximum always sits at an endpoint.

Business Domains Have Edges

Real decisions come with hard limits, and those edges are candidates:

- Quantities are non-negative: $q \geq 0$, the edge $q = 0$ may be optimal (produce nothing)
- Capacity caps output: $q \leq 500$, the unconstrained optimum may lie *past* the limit
- When it does, the best feasible choice is the capacity boundary itself
- Model domain = mathematical domain ∩ business constraints, an intersection, as in Session 2

Your Turn: The Capacity Limit

> **i Question**
>
> You maximise profit and the FOC gives $q = 45$. But your plant can make at most 40 units, so $q \in [0, 40]$ and profit is still *rising* at $q = 40$. What is the best feasible output?

. . .

- The unconstrained optimum $q = 45$ lies outside the feasible interval
- Profit increases all the way up to the edge, so the best feasible choice is the boundary $q = 40$
- Lesson: check the endpoints, a binding capacity limit *is* your optimum

Optimisation in Business

Setting Up a Profit Problem

We reuse Session 4's cost function and add a revenue function:

$$C(q) = 0.5q^2 + 10q + 200 \qquad R(q) = 100q - 0.5q^2$$

- C: fixed costs 200, rising variable costs, marginal cost $C'(q) = q + 10$
- R: revenue peaks then falls, since selling more needs a lower price, $R'(q) = 100 - q$
- Profit is the gap: $\pi(q) = R(q) - C(q)$
- The plan: build π, apply the FOC, verify with the SOC

Worked Example: Build the Profit Function

Subtract cost from revenue, term by term:

$$\pi(q) = (100q - 0.5q^2) - (0.5q^2 + 10q + 200)$$

. . .

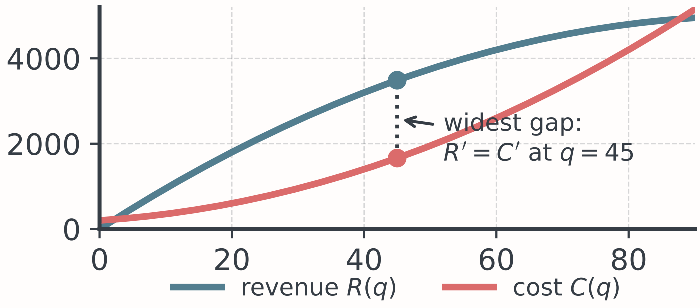
$$\pi(q) = -q^2 + 90q - 200$$

. . .

- A downward parabola, exactly the trade-off shape from Session 3

- Differentiate: $\pi'(q) = -2q + 90$
- The fixed cost 200 vanishes under differentiation, it shifts profit, not the optimum

Solve and Verify



- **FOC:** $\pi'(q) = -2q + 90 = 0 \Rightarrow q = 45$, and $R'(45) = 55 = C'(45)$, marginal revenue = marginal cost
- **SOC:** $\pi''(q) = -2 < 0$ → concave → a maximum, confirmed

Your Turn: Fixed Costs Rise

i Question

The rent goes up: fixed costs rise from 200 to 500, so $C(q) = 0.5q^2 + 10q + 500$. What happens to the optimal output $q = 45$ and to the maximum profit?

(a) q falls, profit falls

(b) q rises, profit falls

(c) q unchanged, profit falls by 300

(d) q unchanged, profit unchanged

...

(c): $\pi(q) = -q^2 + 90q - 500$ has the same derivative $\pi'(q) = -2q + 90$, so the FOC still gives $q = 45$; the constant only shifts the curve down.

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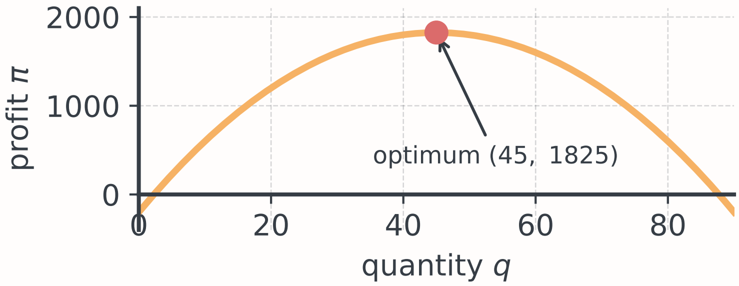
Tip

Fixed costs never change the *decision*, only the *result*. That is the practical meaning of "constants differentiate to zero".

Reading the Optimum



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The optimal quantity is $q = 45$, and the maximum profit is

$$\pi(45) = -45^2 + 90 \cdot 45 - 200 = -2025 + 4050 - 200 = 1825,$$

that is €1825, the highest point of the profit curve.

Your Turn: A Better Price

> **i Question**
>
> Demand improves and revenue becomes $R(q) = 120q - 0.5q^2$, cost stays $C(q) = 0.5q^2 + 10q + 200$. What is the new optimal output? Shout the number.

...

- Profit: $\pi(q) = 120q - 0.5q^2 - 0.5q^2 - 10q - 200 = -q^2 + 110q - 200$
- **FOC:** $\pi'(q) = -2q + 110 = 0 \Rightarrow q = 55$
- **SOC:** $\pi''(q) = -2 < 0$, a maximum; and $R'(55) = 65 = C'(55)$ ✓

Worked Example: Minimising Cost per Unit

A plant's total cost is $C(q) = 400 + 4q + 0.01q^2$. Which output makes the cost per unit smallest?

- Average cost: $A(q) = \frac{C(q)}{q} = \frac{400}{q} + 4 + 0.01q$
- Two opposing forces: the fixed cost spreads thinner ($400/q$ falls), the variable cost per unit grows ($0.01q$ rises)
- Rewrite for the power rule: $A(q) = 400q^{-1} + 4 + 0.01q$

Solve: Where Cost per Unit Bottoms Out

- **FOC:** $A'(q) = -\frac{400}{q^2} + 0.01 = 0 \Rightarrow q^2 = 40000 \Rightarrow q = 200$
- **SOC:** $A''(q) = \frac{800}{q^3} > 0$ for $q > 0$: convex, a minimum
- **Value:** $A(200) = 2 + 4 + 2 = 8$, at 200 units each unit costs €8 on average

...

Notice the shape $\frac{a}{q} + bq$: a falling term against a rising one. The next example, the most famous formula in inventory management, has exactly the same shape.

Logistics: The Economic Order Quantity

A warehouse orders a product with steady annual demand. Order too often and ordering costs pile up; order too much and holding costs pile up. Balance them.

- Annual demand D, fixed cost K per order, holding cost h per unit per year, order size q
- **Ordering cost:** $\frac{D}{q} \cdot K$, $\frac{D}{q}$ orders per year, each costing K
- **Holding cost:** $\frac{q}{2} \cdot h$, average stock is $\frac{q}{2}$ (fill to q, draw down to 0)
- Total: $T(q) = \frac{D}{q}K + \frac{q}{2}h$

Your Turn: Bigger Batches

i Question

You switch to **larger** orders q, same annual demand D. What happens to the two cost terms in $T(q) = \frac{D}{q}K + \frac{q}{2}h$?

(a) Ordering cost rises, holding cost falls

(b) Ordering cost falls, holding cost rises

(c) Both fall

(d) Both rise

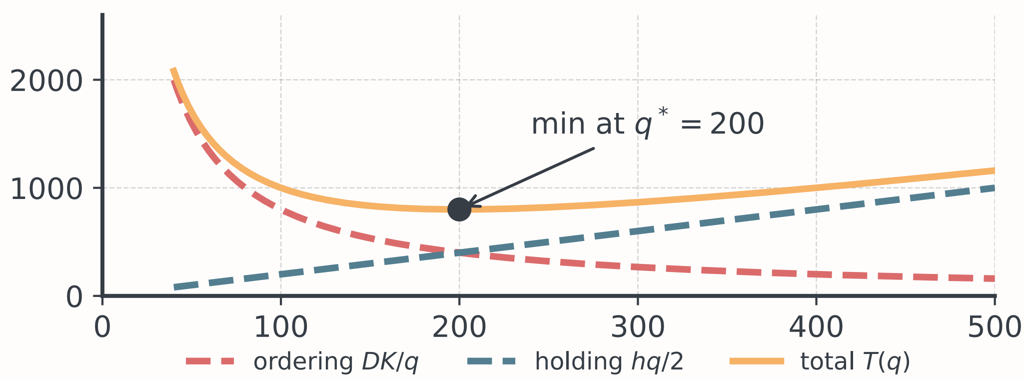
. . .

(b): fewer orders per year (D/q shrinks), but a higher average stock ($q/2$ grows).

. . .

Exactly this tug of war makes the total cost U-shaped, and a U has a bottom to find.

The EOQ: First-Order Condition



With $D = 1000$, $K = 80$, $h = 4$, so $T(q) = 80000/q + 2q$:

- **FOC:** $T'(q) = -80000/q^2 + 2 = 0 \Rightarrow q^2 = 40000 \Rightarrow q^* = 200$

The EOQ: Verify and Read

- **SOC:** $T''(q) = 160000/q^3 > 0$ for all $q > 0$ → convex → a minimum
- At $q^* = 200$: ordering cost $= 400$ and holding cost $= 400$, equal, total €800

The Square-Root Formula

Solve the FOC in symbols and the classic EOQ formula drops out:

$$\frac{DK}{q^2} = \frac{h}{2} \Rightarrow q^2 = \frac{2DK}{h} \Rightarrow \boxed{q^* = \sqrt{\frac{2DK}{h}}}$$

. . .

- Check: $\sqrt{\frac{2 \cdot 1000 \cdot 80}{4}} = \sqrt{40000} = 200$ ✓
- The optimum always balances the two costs, that is *why* they came out equal
- One tidy formula behind textbook inventory management, derived in four lines

Your Turn: Compute the EOQ

> **ℹ Question**
>
> Annual demand $D = 1800$ units, $K = 25$ per order, holding cost $h = 4$ per unit and year. What is the economic order quantity? Shout the number.

. . .

$$q^* = \sqrt{\frac{2 \cdot 1800 \cdot 25}{4}} = \sqrt{\frac{90000}{4}} = \sqrt{22500} = 150$$

. . .

Check the balance: ordering cost $\frac{1800}{150} \cdot 25 = 300$, holding cost $\frac{150}{2} \cdot 4 = 300$, equal, as the theory promised.

From Order Size to Order Frequency

The EOQ also fixes the rhythm of ordering:

- Orders per year: $\frac{D}{q^*} = \frac{1800}{150} = 12$, one order a month
- Time between orders: $\frac{q^*}{D}$ years $= \frac{150}{1800} = \frac{1}{12}$ year, about 30 days
- In our first example ($D = 1000$, $q^* = 200$): 5 orders a year, one every 73 days

. . .

Tip

The formula answers “how much”; dividing by demand answers “how often”. Both are needed for a replenishment plan.

Your Turn: Ordering Cost Doubles

Question

A supplier doubles the fixed cost K per order. According to $q^* = \sqrt{\frac{2DK}{h}}$, does the economic order quantity rise or fall, and by how much?

. . .

- q^* grows with K under the root, so a larger K makes q^* rise
- Doubling K multiplies q^* by $\sqrt{2} \approx 1.41$, about a 41% larger order
- Intuition: if each order costs more, you order less often in bigger batches

Closing

Key Takeaways

- **First-order condition:** optima hide where $f'(x) = 0$, solve it for the candidates
- **Second-order condition:** $f''(x^*) > 0$ a minimum, $f''(x^*) < 0$ a maximum, $f'' = 0$ inconclusive, the FOC alone is never enough, since a flat tangent can be a saddle (x^3 at 0)
- On a closed interval, compare stationary points **and** endpoints, boundaries can win
- Profit is maximal where marginal revenue = marginal cost, Session 4's promise, delivered
- The EOQ $q^* = \sqrt{2DK/h}$ balances ordering against holding costs

Skip order if running long: 1. Reading the Optimum plot (state $\pi(45) = 1825$ verbally on the previous slide), 2. The first-derivative-test bullets on the “FOC Is Never the Whole Story” slide, 3. Business Domains Have Edges (fold into the Your Turn capacity slide), 4. compress the R & C plot to a single sentence about the widest gap.

That’s it for today.

Any Questions?

Until the Next Session

- Work through the Tasks: optimisation drills with worked solutions
- Check yourself with the Self-Test quiz
- Keep the Cheatsheet next to you, the FOC/SOC recipe lives on it
- Note down anything unclear, we start next session with **your questions**

...

> Tip
>
> The recipe is always the same: differentiate, set $f' = 0$, solve, then let f'' decide. Name the goal (maximise or minimise) and the sign of f'' you want follows.

Preview: Session 06: Systems of Linear Equations

- Central question: solve for several unknowns at once
- From one equation to a whole system, where do the lines meet?
- Elimination and substitution, done systematically
- Gaussian elimination for bigger systems, market equilibrium, production planning

...

See you there, and bring your questions!

Literature & Further Reading

- These sessions cover the essentials: textbooks offer more depth and practice
- Start with K. Sydsæter, P. J. Hammond, and A. Strøm [1] or I. Jacques [2]; full recommendations on the tutorial's literature page

Bibliography

[1] K. Sydsæter, P. J. Hammond, and A. Strøm, *Essential Mathematics for Economic Analysis*, 4th ed. Harlow: Pearson, 2012.

[2] I. Jacques, *Mathematics for Economics and Business*, 8th ed. in Always Learning. Harlow: Pearson, 2015, p. 660.