Session 03 - Functions

Mathematics for Master Students

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Recap & Your Questions

Session 02 in a Nutshell

- A set is an unordered collection of distinct elements: $\in$ for elements, $\subseteq$ for sets
- $\cup$, $\cap$, $\setminus$, ^{c} mirror **or**, **and**, **not**: De Morgan flips $\cup$ and $\cap$
- Counting a union: $|A \cup B| = |A| + |B| - |A \cap B|$
- $A \times B$ collects ordered pairs: $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the plane
- Constraints are sets, and they combine by **intersection**
- Today: functions map between sets, and their graphs live in $\mathbb{R}^2$

Common Issues in the Homework

- **Element vs subset:** braces build a *set*: $\{2\} \subseteq A$ pairs with $\subseteq$, while $2 \in A$ pairs with $\in$; mixing them was the most frequent slip
- **Complements reverse inclusion:** from $A \subseteq B$ it follows that $B^c \subseteq A^c$: writing $A^c \subseteq B^c$ is the classic inversion error
- **Mixed $\cup$ and $\cap$:** $A \cap B \cup C$ is ambiguous: grouping changes the set, so always add parentheses

. . .

> Tip
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> Which tasks gave you trouble? Bring them up **now**: we take the time to go through them before we move on to today's topic.

Warm-Up: One Quick Evaluation

> Question
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> Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$. Compute $(A \cup B) \setminus (A \cap B)$.

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- Inside out: $A \cup B = \{1, 2, 3, 4, 5\}$ and $A \cap B = \{3, 4\}$
- Removing the intersection: $(A \cup B) \setminus (A \cap B) = \{1, 2, 5\}$
- In words: everything in exactly one of the two sets: “either, but not both”

Today's Plan

- **What is a function?**: exactly one output for every input
- **Function gallery**: linear, quadratic, polynomial, rational, exponential, logarithm
- **Domain in practice**: which inputs are allowed
- **Combining functions**: composition and inverses
- **Properties**: monotone, bounded, convex vs concave

Functions as Mappings

What Is a Function?

A function $f : A \to B$ is a **rule** that assigns to each element $x \in A$ exactly one element $f(x) \in B$.

- It maps **between two sets**: the ones from last session
- Think of a *machine*: input x goes in, output $f(x)$ comes out
- "Exactly one" makes it predictable: same input, same output, every time

...

⚠ Common Mistake

"Exactly one" restricts the **outputs per input**. Two different inputs may share the same output: that is perfectly fine.

Function Notation

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 2x + 5$$

- f is the function's **name**; x is the input, also called the argument
- $f(x)$ is the output: read "f of x"; here $f(x) = 2x + 5$
- $x \mapsto f(x)$ reads "x maps to $f(x)$": note the little bar on the arrow
- $f : A \to B$ names the two sets: inputs come from A, outputs land in B

...

Tip

f and $f(x)$ are different things: f is the **rule**, $f(x)$ is a **number**: the value at x.

Domain, Codomain, Range

For $f : A \to B$, three sets matter:

- The domain A: all **allowed inputs**
- The codomain B: where outputs are *declared* to land
- The range $f(A) = \{f(x) : x \in A\}$: the outputs that **actually occur**
- Always $f(A) \subseteq B$: the range is a subset of the codomain

. . .

Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$: codomain $\mathbb{R}$, but range $[0, \infty)$: no square is negative.

Your Turn: What Is the Range?

ℹ Question

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 1$. Which set is the **range** of f?

(a) $\mathbb{R}$ **(b)** $[0, \infty)$

(c) $[1, \infty)$ **(d)** $(1, \infty)$

. . .

(c): $x^2 \geq 0$, so $x^2 + 1 \geq 1$, and the value 1 is reached at $x = 0$: square bracket.

. . .

(a) is the codomain, not the range: the declared landing zone is larger than the outputs that actually occur.

Evaluating a Function

To evaluate, substitute the input everywhere the variable appears, take $f(x) = x^2 - 2x$:

- $f(3) = 3^2 - 2 \cdot 3 = 9 - 6 = 3$
- $f(0) = 0 - 0 = 0$
- $f(-1) = (-1)^2 - 2 \cdot (-1) = 1 + 2 = 3$
- Note: $f(3) = f(-1) = 3$: two inputs, same output, allowed

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Tip

Put negative inputs in **parentheses** before squaring: $(-1)^2 = 1$, not -1.

Plugging in Whole Expressions

The input can be an expression: substitute all of it, in parentheses. Again $f(x) = x^2 - 2x$:

$$f(a + h) = (a + h)^2 - 2(a + h) = a^2 + 2ah + h^2 - 2a - 2h$$

. . .

Common Mistake

$f(a+h) \neq f(a) + f(h)$, check: $f(1+1) = f(2) = 0$, but $f(1) + f(1) = -1 - 1 = -2$.

. . .

Expressions like $f(a+h)$ are the raw material of **Session 4**: "how much does f change when the input moves by h?"

Your Turn: Evaluate

Question

Again $f(x) = x^2 - 2x$. What is $f(-2)$? Type the number. Bonus: simplify $f(x+1)$.

. . .

$$f(-2) = (-2)^2 - 2 \cdot (-2) = 4 + 4 = 8$$

. . .

$$f(x+1) = (x+1)^2 - 2(x+1) = x^2 + 2x + 1 - 2x - 2 = x^2 - 1$$

. . .

Common Mistake

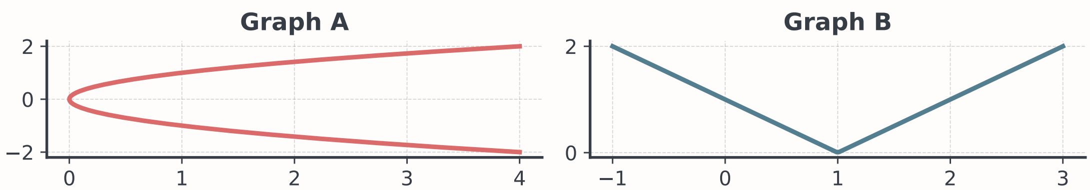
$f(-2) = 0$ comes from dropping a sign: either $(-2)^2 = -4$ or $-2 \cdot (-2) = -4$. Both minus signs matter.

The Vertical Line Test

The graph of f is the set $\{(x, f(x)) : x \in A\} \subseteq \mathbb{R}^2$: ordered pairs, as in Session 2.

- A curve is the graph of a function $\Leftrightarrow$ every vertical line hits it **at most once**
- Why: one input x must yield exactly one output
- The circle $x^2 + y^2 = 1$ fails: at $x = 0$ it contains *two* points, $y = 1$ and $y = -1$
- Lines, parabolas and exponential curves all pass

Your Turn: Which Graph Is a Function?



Question

Which of the two graphs shows y as a function of x?

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Only Graph B: A fails the vertical line test: $x = 1$ gives both $y = 1$ and $y = -1$. The kink in B is fine.

Functions in Business

Functions map decisions to outcomes, the core of every quantitative model:

- **Cost function** $C(q)$: production quantity $\to$ total cost
- **Demand function** $D(p)$: price $\to$ units sold
- **Production function**: input (labour hours) $\to$ output (units)
- One decision in, one predicted outcome out: that is why models must be functions

...

Tip

Whenever a spreadsheet column is computed from another column, there is a function behind it.

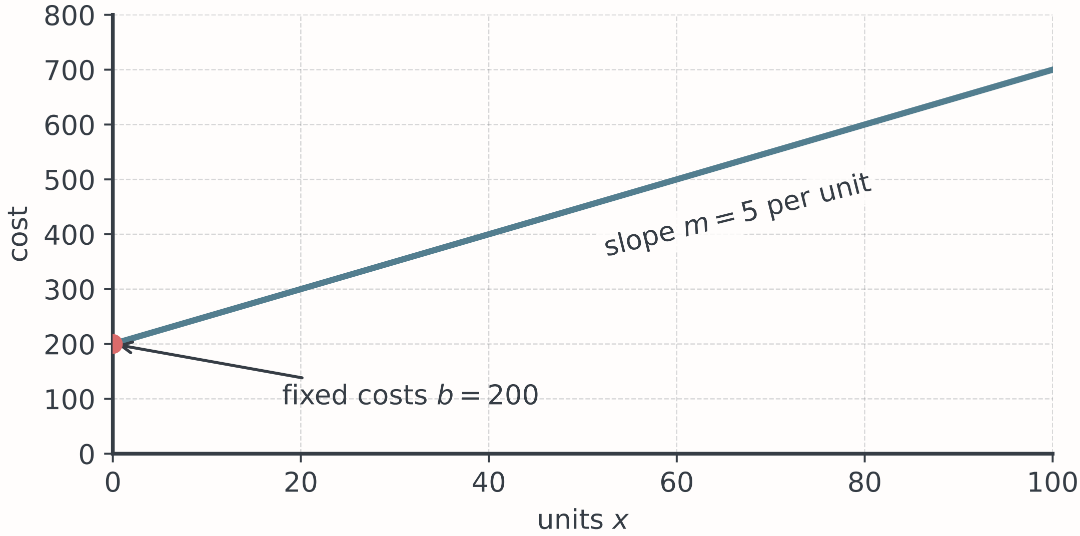
Function Gallery

Linear Functions

$$f(x) = mx + b$$

- Slope m: the output changes by m for **each unit** of input, constant everywhere
- Intercept $b = f(0)$: the starting level
- From two points on the graph: $m = \frac{f(x_2)-f(x_1)}{x_2-x_1}$
- The workhorse of business models: constant rates

A Linear Cost Function



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$C(x) = 200 + 5x$: the fixed costs are the intercept, the variable cost per unit is the slope.

Your Turn: Slope from Two Points

Question

A linear cost function gives $C(10) = 250$ and $C(30) = 350$. What are the slope m and the intercept b? Type both numbers.

...

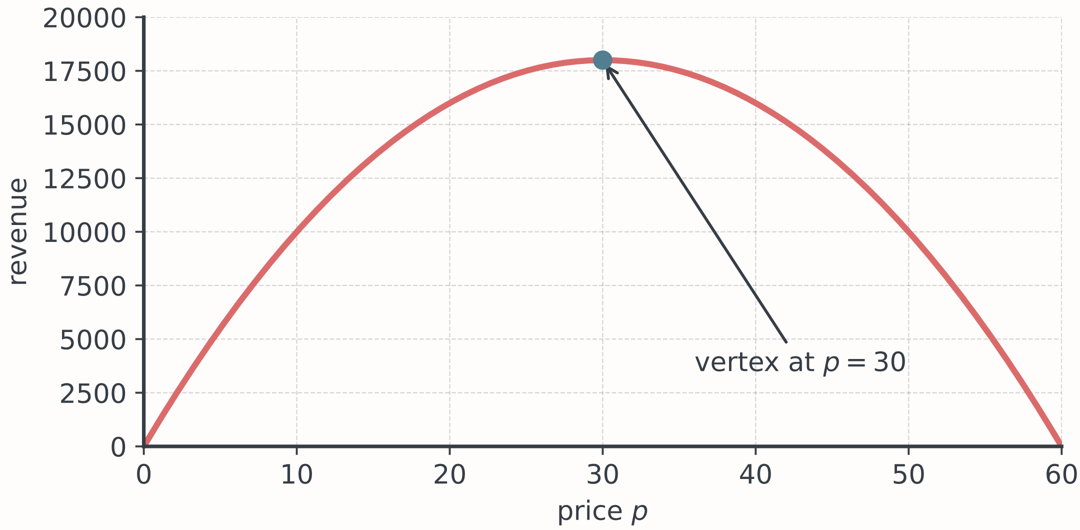
- Slope: $m = \frac{350-250}{30-10} = \frac{100}{20} = 5$, €5 of variable cost per unit
- Intercept: $b = C(10) - 5 \cdot 10 = 250 - 50 = 200$, the fixed costs
- So $C(x) = 200 + 5x$: the function from the plot, recovered from two data points

Quadratic Functions

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

- The graph is a parabola: opens **upward** if $a > 0$, **downward** if $a < 0$
- The turning point is the vertex, located at $x = -\frac{b}{2a}$
- Downward parabolas model **trade-offs**: revenue rises with price, until demand collapses

A Quadratic Revenue Function



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Demand $D(p) = 1200 - 20p$ gives revenue $R(p) = p \cdot D(p) = 1200p - 20p^2$: the vertex $p = -1200/(2 \cdot (-20)) = 30$ is the revenue-maximal price.

Your Turn: Where Is the Vertex?

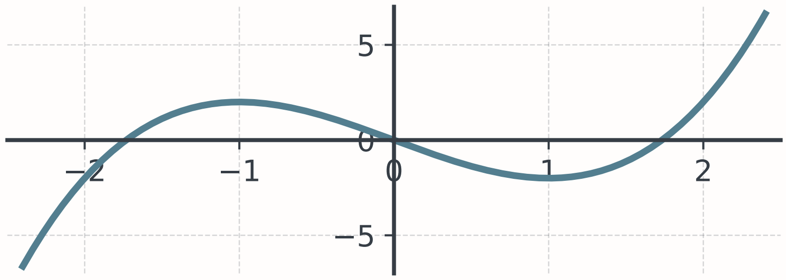
> **Question**
>
> Another product has revenue $R(p) = 800p - 10p^2$. Does the parabola open upward or downward, and at which price is the vertex? Type the number.

...

- $a = -10 < 0$: the parabola opens **downward**, the vertex is the revenue peak
- Vertex at $p = -\frac{b}{2a} = -\frac{800}{2\cdot(-10)} = 40$
- Check: the demand behind it is $D(p) = 800 - 10p$, which hits zero at $p = 80$, and the peak sits halfway

Polynomials

$$f(x) = a_n x^n + \ldots + a_1 x + a_0 \qquad \text{degree} \quad n = \text{highest power}$$



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Degree n allows up to n roots and up to $n-1$ turning points: far from the origin the leading term $a_n x^n$ dominates ($f(x) = x^3 - 3x$ above behaves like x^3).

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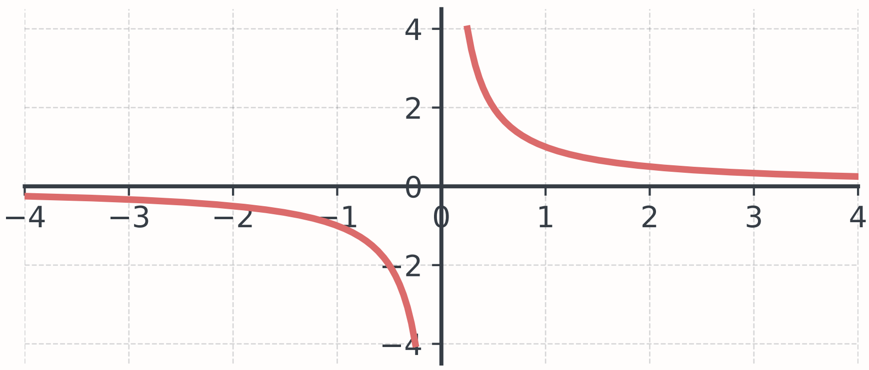
Cubic cost curves $C(x) = ax^3 + bx^2 + cx + d$ are the classic economics example: costs rise steeply, flatten, then steepen again.

Rational Functions

$$f(x) = \frac{p(x)}{q(x)} \qquad \text{a quotient of two polynomials}$$

- Defined only where $q(x) \neq 0$: division by zero is the first domain trap
- Near a zero of the denominator, values explode: a **vertical asymptote**
- For large inputs, the graph can settle toward a **horizontal asymptote**

The Hyperbola $f(x) = 1/x$



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Both axes are asymptotes. **Business example:** with $C(x) = 200 + 5x$, the *cost per unit* is $C(x)/x = 200/x + 5$: it falls toward 5 as volume grows: economies of scale.

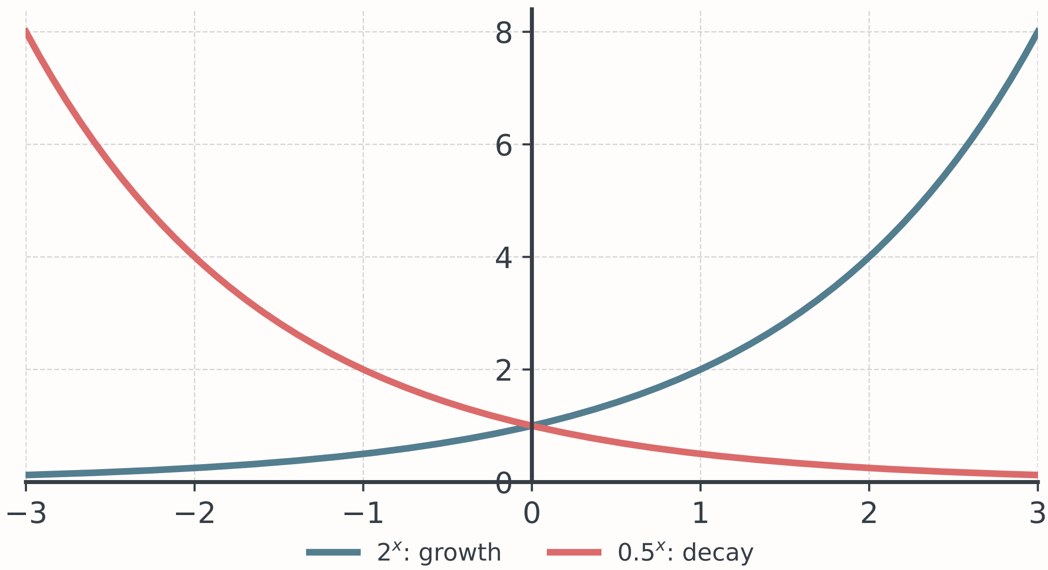
Exponential Functions

$$f(x) = a \cdot b^x, \quad b > 0$$

- The **variable sits in the exponent**: each step multiplies by b

- $b > 1$: growth (interest, booming demand); $0 < b < 1$: decay (depreciation, churn)
- $a = f(0)$ is the starting value
- The standard base is Euler's number $e \approx 2.71828$: Sessions 4–5 show why it is so convenient

Growth and Decay



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Both stay strictly positive and pass through $(0, 1)$: an exponential never reaches zero.

Your Turn: Growth or Decay?

i Question

A delivery van's resale value follows $V(t) = 50000 \cdot 0.8^t$ (in €, after t years). Which statement is true?

(a) The value grows by 20% per year

(b) The value shrinks by 20% per year

(c) The value shrinks by €10000 per year

(d) The value reaches €0 after 5 years

...

(b): base $0.8 < 1$ means decay, and each year keeps 80% of the previous value.

...

⚠ Common Mistake

(c) is the *first* year only: $50000 \to 40000$. The second year loses 8000, the third 6400. And (d) never happens: an exponential stays strictly positive.

Compound Interest

Deposit K_0 at annual rate r, each year multiplies the balance by $(1+r)$:

$$K_t = K_0 \cdot \underbrace{(1+r)(1+r)\ldots(1+r)}_{t \text{ factors}} = K_0 \cdot (1+r)^t$$

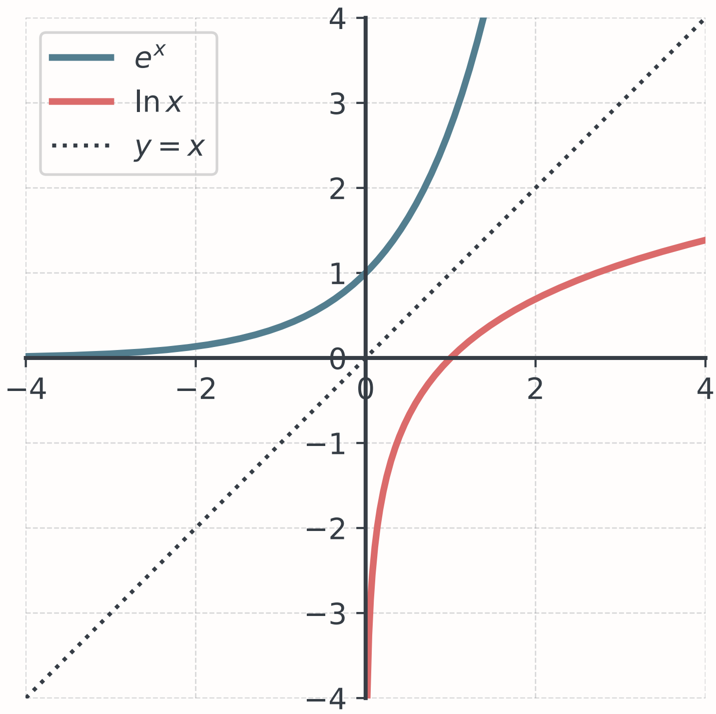
- Session 1's product idea in action: repeated multiplication becomes a power
- €1000 at $r = 0.05$: after 15 years $1000 \cdot 1.05^{15} \approx$ €2079: more than **doubled**
- As a function of time t, this is exponential growth with base $b = 1.05$

Logarithms: Undoing e^x

The natural logarithm $\ln x$ answers: *which exponent produces x?*

$$y = \ln x \Leftrightarrow x = e^y$$

- $\ln x$ is the mirror image of e^x across the diagonal
- Defined only for $x > 0$
- $\ln 1 = 0$ and $\ln e = 1$



Rules for Logarithms

$$\ln(xy) = \ln x + \ln y \qquad \ln\frac{x}{y} = \ln x - \ln y \qquad \ln(x^k) = k\ln x$$

- Logs turn **products into sums** and powers into multiples
- The power rule pulls the unknown **down from the exponent**: that is why logs solve growth questions
- Other bases reduce to $\ln$: $\log_b x = \frac{\ln x}{\ln b}$

. . .

⚠ Common Mistake

$\ln(x+y) \neq \ln x + \ln y$: the rules work for *products*, not sums.

Your Turn: Which Log Rule?

ℹ Question

Which expression equals $\ln(x^2 y)$ for $x, y > 0$?

(a) $2\ln x + \ln y$ **(b)** $2\ln x \cdot \ln y$
(c) $\ln(x^2) \cdot \ln y$ **(d)** $2\,(\ln x + \ln y)$

. . .

(a): the product rule splits $\ln(x^2 y) = \ln(x^2) + \ln y$, then the power rule gives $\ln(x^2) = 2\ln x$.

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(d) would be $\ln(x^2 y^2)$: the factor 2 belongs to x only, so apply the power rule *after* splitting the product.

Solving for the Exponent

When the unknown sits in the exponent, take $\ln$ on both sides and pull it down.

Question: a warehouse's throughput grows 8% per year. After how many years has it **tripled**?

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- Model: $1.08^t = 3$
- Take logs: $\ln(1.08^t) = \ln 3$, so $t\ln 1.08 = \ln 3$
- Solve: $t = \frac{\ln 3}{\ln 1.08} \approx \frac{1.0986}{0.0770} \approx 14.3$ years

. . .

The same three steps handle *every* "how long until" question with exponential growth.

Your Turn: When Has It Doubled?

> ℹ Question
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> With continuous growth at rate r, capital follows $K_t = K_0 \cdot e^{rt}$. When has it doubled?

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$$e^{rt} = 2 \quad \Rightarrow \quad rt = \ln 2 \quad \Rightarrow \quad t = \frac{\ln 2}{r} \approx \frac{0.693}{r}$$

- At $r = 5\%$: $t \approx \frac{0.693}{0.05} \approx 13.9$ years: independent of the starting amount
- The same trick handles yearly compounding: $1.05^t = 2$ gives $t = \frac{\ln 2}{\ln 1.05} \approx 14.2$ years

Domain in Practice

The Maximal Domain

If no domain is stated, take the maximal domain: the largest set of reals on which the formula is defined.

Three red flags:

- **Division by zero**: denominators must stay $\neq 0$
- **Even roots of negatives**: under $\sqrt{\ }$ we need ≥ 0
- **Logs of non-positives**: inside $\ln$ we need > 0 (strictly!)

...

Everything else (polynomials, exponentials) is safe on all of $\mathbb{R}$.

Finding the Domain: Examples

Function	Restriction	Maximal domain
$\frac{1}{x-3}$	$x - 3 \neq 0$	$\mathbb{R} \setminus \{3\}$
$\sqrt{2x+6}$	$2x + 6 \geq 0$	$[-3, \infty)$
$\ln(5 - x)$	$5 - x > 0$	$(-\infty, 5)$

...

> Tip
>
> The results are **intervals**. Session 1's notation pays off: square bracket at -3 ($\sqrt{0}$ is fine), round bracket at 5 ($\ln 0$ is not).

Your Turn: Find the Domain

Question

What is the maximal domain of $f(x) = \frac{\sqrt{x-2}}{x-4}$?

...

- The root needs $x - 2 \geq 0$, so $x \geq 2$
- The denominator needs $x - 4 \neq 0$, so $x \neq 4$
- **Both** must hold, intersect the conditions: $[2, 4) \cup (4, \infty)$

...

Tip

Collect *every* restriction first, then combine: one forgotten condition is the typical exam slip.

Domains in Business

Business adds its own restrictions on top of the maths:

- Quantities are **non-negative**: $q \in [0, \infty)$: no producing -50 pallets
- Capacity caps the allowed quantities: $q \in [0, 500]$
- Prices must keep demand meaningful: $D(p) = 1200 - 20p \geq 0$ forces $p \in [0, 60]$
- Model domain = mathematical domain ∩ business constraints: an intersection, as in Session 2

Combining Functions

Composition of Functions

Chaining two functions, apply g first, then f to the result:

$$(f \circ g)(x) = f(g(x))$$

- Read from the inside out: the inner function acts first
- **Example:** $g(x) = 2x$ and $f(x) = x + 3$: $(f \circ g)(x) = f(2x) = 2x + 3$
- But $(g \circ f)(x) = g(x + 3) = 2x + 6$: not the same
- Business reading: multi-stage processes, the output of one stage feeds the next

Order Matters: Voucher and VAT

A webshop applies a €10 voucher $v(p) = p - 10$ and 19% VAT $t(p) = 1.19p$ to a net price p:

- **Voucher first, then tax:** $(t \circ v)(p) = 1.19(p - 10) = 1.19p - 11.90$
- **Tax first, then voucher:** $(v \circ t)(p) = 1.19p - 10$

- The results differ by €1.90 on *every* order: for the customer, voucher first is cheaper

. . .

> ⚠ Common Mistake
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> Assuming $f \circ g = g \circ f$: in general, the order cannot be swapped. Always check which function acts first.

Your Turn: Compose in the Right Order

> ℹ Question
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> Let $f(x) = x^2$ and $g(x) = x + 1$. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.

. . .

- $(f \circ g)(2) = f(g(2)) = f(3) = 9$
- $(g \circ f)(2) = g(f(2)) = g(4) = 5$
- Inside out, always, and the order visibly changes the answer

Inverse Functions

The inverse f^{-1} **undoes** f:

$$f(a) = b \quad \Leftrightarrow \quad f^{-1}(b) = a$$

- It exists only if f is one-to-one: every output comes from **exactly one** input
- Graph check: every horizontal line hits the graph at most once
- $f(x) = x^2$ on $\mathbb{R}$ fails: $f(2) = f(-2) = 4$: which input produced 4?
- Restricting to $[0, \infty)$ repairs it: there $f^{-1}(x) = \sqrt{x}$

Finding an Inverse

Write $y = f(x)$, solve for x, then read the result as a function of the output.

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Demand example: $q = D(p) = 1200 - 20p$:

$$q = 1200 - 20p \quad \Rightarrow \quad 20p = 1200 - q \quad \Rightarrow \quad p = 60 - \frac{q}{20}$$

. . .

- $D^{-1}(q) = 60 - \frac{q}{20}$: the price needed to sell exactly q units
- Check with a pair: $D(30) = 600$ and $D^{-1}(600) = 60 - 30 = 30$ ✓
- Domain and range swap: the domain of D^{-1} is the range of D

Your Turn: Undo the Function

Question

Let $f(x) = 3x - 6$. For which x is $f(x) = 9$, i.e. what is $f^{-1}(9)$? Type the number.

...

- Solve $3x - 6 = 9$: $3x = 15$, so $x = 5$
- In general: $y = 3x - 6 \Rightarrow x = \frac{y+6}{3}$, hence $f^{-1}(y) = \frac{y+6}{3}$
- Check: $f^{-1}(9) = \frac{15}{3} = 5$ ✓

The Notation Trap: $f^{-1} \neq 1/f$

- $f^{-1}(x)$ is the **inverse function**: the -1 sits on the function *name*
- The reciprocal of the *value* is written $(f(x))^{-1} = \frac{1}{f(x)}$
- **Example** $f(x) = 2x$: $f^{-1}(x) = \frac{x}{2}$, but $\frac{1}{f(x)} = \frac{1}{2x}$: completely different

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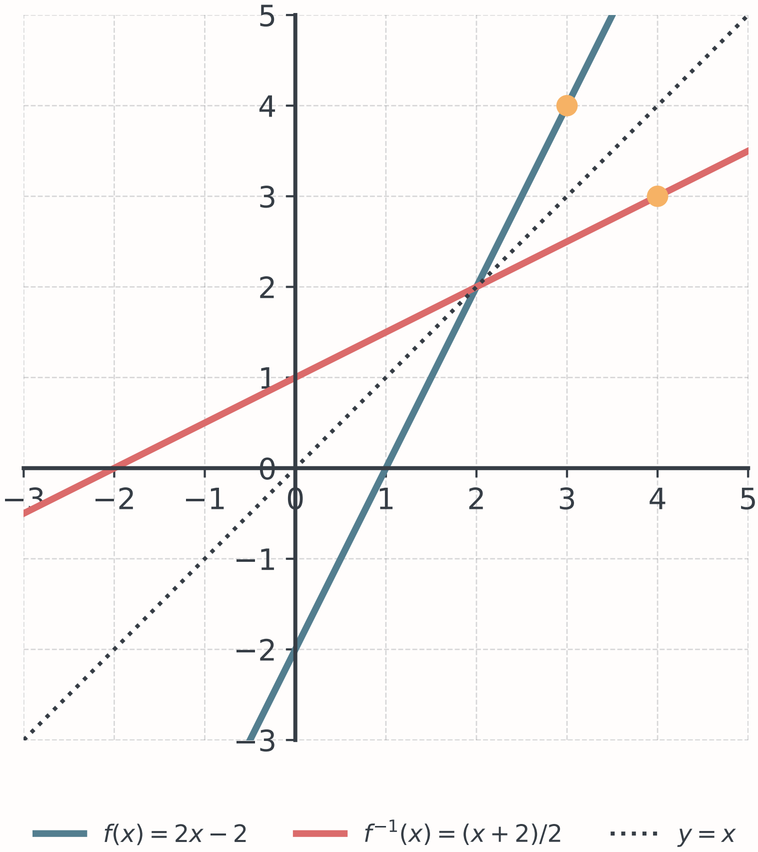
Common Mistake

Reading f^{-1} as "one over f": for functions, the exponent -1 means undo, not divide.

The Mirror Property



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Swapping input and output reflects the graph across $y = x$: the point $(3, 4)$ on f becomes $(4, 3)$ on f^{-1}: exactly how $\ln x$ mirrored e^x.

Properties

Monotonicity

f is increasing if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$: **strictly** increasing with $<$. Decreasing works the same way, flipped.

- Graph reading: strictly increasing = rises left to right, everywhere
- Costs increase in quantity; demand decreases in price
- Strictly monotone $\Rightarrow$ one-to-one $\Rightarrow$ invertible: that is why e^x has an inverse but x^2 on $\mathbb{R}$ does not

Your Turn: Invertible or Not?

i Question

Which of these functions is **not** one-to-one on $\mathbb{R}$, and therefore has no inverse there?

(a) $f(x) = 2x + 1$ **(b)** $f(x) = e^x$
(c) $f(x) = x^2 - 4$ **(d)** $f(x) = x^3$

...

(c): $f(2) = f(-2) = 0$, one output from two inputs, so it fails the horizontal line test.

...

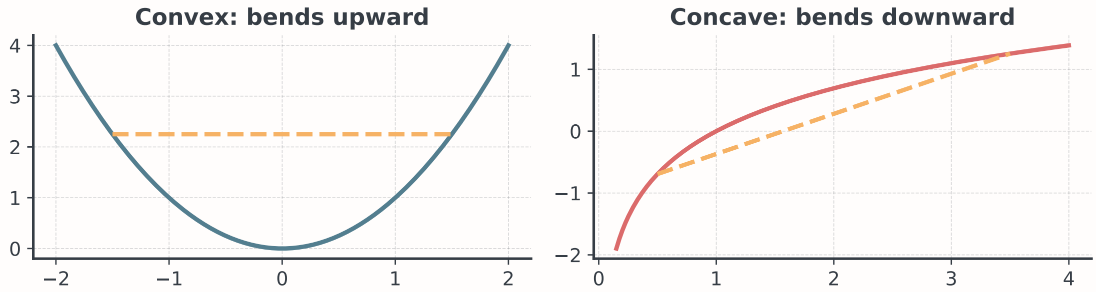
The other three are strictly increasing on all of $\mathbb{R}$, and strictly monotone functions are always invertible.

Boundedness

f is bounded above if some M satisfies $f(x) \leq M$ for **all** x: bounded below works with $\geq m$.

- $e^x > 0$: bounded **below** by 0, unbounded above
- A market share lives in $[0, 1]$: bounded on both sides
- $f(x) = x^2$: bounded below by 0, but grows without limit
- Bounds tell you what a model can, and cannot, predict

Convex or Concave?



- Convex: for two graph points, the chord lies **above the graph**: x^2, e^x
- Concave: the chord lies **below the graph**: $\ln x$, $\sqrt{x}$: *diminishing returns*
- Intuition only for now: Sessions 4–5 make it precise and find optima

Your Turn: Chord Check

> **i Question**
>
> Which of these functions is **concave** on its domain?
>
> **(a)** x^2 **(b)** e^x
>
> **(c)** $\sqrt{x}$ **(d)** $|x|$

...

(c): $\sqrt{x}$ rises ever more slowly, its chords lie below the graph: diminishing returns.

...

(a) and (b) bend upward, and the chords of $|x|$ also lie above its graph: all three are convex. Session 4 turns the chord test into a sign test on f''.

Closing

Key Takeaways

- A function assigns each input exactly one output: domain, codomain and range are *sets*
- Know the gallery shapes: linear, quadratic, polynomial, rational, exponential, logarithm
- $\ln x$ undoes e^x: the power rule $\ln(x^k) = k \ln x$ solves for exponents
- Maximal domain: no division by zero, no even roots of negatives, no logs of non-positives
- Composition chains processes: order matters; inverses undo, but only one-to-one functions have them
- Monotone, bounded, convex/concave: reading **behaviour** from a graph

Skip order if running long: 1. Mirror Property, 2. Compound Interest (keep one bullet on Growth & Decay), 3. Notation Trap (compress into Inverse Functions callout), 4. compress Functions in Business to its punchline.

That's it for today.

Any Questions?

Until the Next Session

- Work through the Tasks: problems with worked solutions
- Check yourself with the Self-Test quiz
- Keep the Cheatsheet next to you while practising
- Note down anything unclear: we start next session with **your questions**

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> 💡 Tip
>
> When a function confuses you, **sketch it**: five plotted points reveal more than ten minutes of staring at the formula.

Preview: Session 04: Differentiation

- Central question: how fast does a function change?
- Slopes of *curved* graphs: the tangent line
- Today's $f(a+h)$ becomes the **difference quotient**
- Marginal cost and marginal revenue: the language of economic decisions
- And e^x reveals its superpower

. . .

See you there, and bring your questions!

Literature & Further Reading

- These sessions cover the essentials: textbooks offer more depth and practice
- Start with K. Sydsæter, P. J. Hammond, and A. Strøm [1] or I. Jacques [2]; full recommendations on the tutorial's literature page

Bibliography

[1] K. Sydsæter, P. J. Hammond, and A. Strøm, *Essential Mathematics for Economic Analysis*, 4th ed. Harlow: Pearson, 2012.

[2] I. Jacques, *Mathematics for Economics and Business*, 8th ed. in Always Learning. Harlow: Pearson, 2015, p. 660.