Cheatsheet 06 - Systems of Linear Equations

Mathematics for Master Students

What "Linear" Means

An equation is **linear** when every unknown appears on its own, to the first power: a weighted sum of the variables equals a number. No powers, no products of variables, no roots or exponents on a variable.

Linear	Not linear
$2x + 3y = 13$	$x^2 + y = 4$ (a square)
$x_1 + 2x_2 + 2x_3 = 16$	$xy = 6$ (a product)
$4a + 7b - c = 0$	$\sqrt{x} + y = 1$ (a root)

A **system** asks several linear equations to hold at the same time. A **solution** is a set of values satisfying **every** equation at once. Always check a candidate against all of them.

Solving a 2×2 System

Substitution. Solve one equation for one variable, drop it into the other:

1. Isolate a variable, e.g. $x = 5 - y$.
2. Substitute into the other equation and solve for the remaining variable.
3. Back-substitute to get the first variable.

Elimination. Scale the equations so a variable cancels:

1. Multiply one (or both) equations so a variable has matching coefficients.
2. Add or subtract to cancel that variable; solve for what remains.
3. Back-substitute for the other variable.

Both methods give the **same** point where the two lines cross. Always verify in the original equations.

The Augmented Matrix

A system is just its numbers. Stack the coefficients, then the right-hand side after a bar. For example, $x + y + z = 6,\ x + 2y + 3z = 11,\ 2x + y + z = 9$ becomes

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 11 \\ 2 & 1 & 1 & 9 \end{bmatrix}$$

Each row is an equation; the bar separates left from right. No matrix algebra: pure bookkeeping.

> The Three Row Operations
>
> Each one keeps the solution set unchanged:
>
> 1. **Swap** two rows (reorder the equations).
> 2. **Scale** a row by a **non-zero** number (multiply or divide an equation).
> 3. **Add** a multiple of one row to another (combine two equations).
>
> Every operation acts on the **whole row (the right-hand side included)**.

Gaussian Elimination Recipe

To solve a 3×3 system:

1. **Forward-eliminate**: using row operations, create zeros below the diagonal, one column at a time, until the matrix is **triangular** (a staircase of zeros in the lower-left).
2. **Read the triangular system** from the bottom up: the last row gives one variable directly.
3. **Back-substitute** upward, one variable at a time.
4. **Verify** the triple in the original equations.

Micro-example (from the matrix above):

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow z = 2,\ y + 2(2) = 5 \Rightarrow y = 1,\ x + 1 + 2 = 6 \Rightarrow x = 3.$$

How Many Solutions?

The triangular form tells you the fate without a picture:

In the matrix you see	Number of solutions	Geometry (2 unknowns)
A leading entry in every column	**exactly one**	lines cross at a point
A row $0 = (\text{non-zero})$, e.g. $0 = 5$	**none**	parallel, distinct lines
A row $0 = 0$ (whole row vanishes)	**infinitely many**	the same line

> Read the Right-Hand Side
>
> $0 = 0$ means an equation was **redundant** → *infinitely many* solutions. "No solution" appears only as $0 = (\text{non-zero})$, a true contradiction. Never confuse the two.

Parameterise the free case: when a row vanishes, set the free variable to t. For $x - 3y = 2$ with a redundant partner: let $y = t$, then $x = 2 + 3t$, giving the solution set $(x, y) = (2 + 3t, t)$.

Market Equilibrium

A market **clears where supply equals demand**, one price and quantity solving both curves:

$$\text{demand} \quad q = 90 - 3p, \quad \text{supply} \quad q = 2p - 10 \Rightarrow 90 - 3p = 2p - 10 \Rightarrow p = 20,\ q = 30.$$

Set the two expressions for q equal (or eliminate q from the system), solve for p, then back-substitute for q. Below equilibrium there is excess demand (shortage); above it, excess supply (surplus). The same idea drives production-planning systems.

Systems Checklist

> Before You Answer
>
> - Is every equation actually **linear**? (first power, no products or roots)
> - Did you apply each row operation to the **whole row**, right-hand side included?
> - Did you reduce to **triangular** form before back-substituting?
> - Did you read $0 = 0$ (infinitely many) vs $0 = (\text{non-zero})$ (none) correctly?
> - Did you **verify** the solution in the original equations?
>
> This is the last cheatsheet. Bring all six to the mock exam in our final session.