Cheatsheet 05 - Single Variable Optimization

Mathematics for Master Students

The Optimisation Recipe

To find the optima of a smooth function f, follow four steps:

1. **Differentiate**: compute $f'(x)$.
2. **Solve the FOC** $f'(x) = 0$: the roots are the **stationary points** (the candidates). Factor f' where possible; its roots hand them to you.
3. **Classify** each candidate with the **SOC** $f''(x^*)$ (table below).
4. **Decide global**: on a closed interval, also test the endpoints; on an unbounded domain, check the behaviour as $x \to \pm\infty$.

First-order condition (FOC): every smooth interior extremum satisfies

$$f'(x^*) = 0.$$

It is **necessary but not sufficient**: a flat tangent flags a candidate, nothing more.

Classifying: the Second-Order Condition

At a stationary point x^* (where $f'(x^*) = 0$), curvature decides the type:

$f''(x^*)$	Curvature	x^* is a
> 0	convex (bends up)	**local minimum**
< 0	concave (bends down)	**local maximum**
$= 0$	flat	**inconclusive**: test fails

- Mnemonic: **positive** f'' holds water → **minimum**; **negative** f'' spills it → **maximum**.
- **When $f'' = 0$**, fall back to the **first-derivative test**: if f' changes $+ \to -$ across x^* it is a maximum; $- \to +$ a minimum; **same sign on both sides** → neither (a saddle).

> ⚠ The Saddle Trap
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> $f'(x^*) = 0$ alone never proves an extremum. For $f(x) = x^3$ at $x = 0$: $f'(0) = 0$ **and** $f''(0) = 0$, yet the graph keeps rising: a **saddle point**, neither peak nor valley. Always finish with the SOC (or the first-derivative test).

Convexity and Inflection Points

Read curvature from f'' over a whole interval, not just at a point:

Condition on an interval	Shape
$f''(x) \geq 0$	**convex** (bends up)
$f''(x) \leq 0$	**concave** (bends down)

- Where f'' **changes sign**, the bend switches: an **inflection point**.
- An inflection point is *not* an extremum: f' need not be zero there.
- Example: $f(x) = x^3 - 6x^2 + 5$ has $f''(x) = 6(x - 2)$, concave for $x < 2$, convex for $x > 2$, inflection at $x = 2$.

Local vs Global & Closed Intervals

- A **local** optimum is best only in its neighbourhood; a **global** optimum is best on the whole domain.
- **Extreme Value Theorem:** a continuous function on a *closed* interval $[a, b]$ always attains a global max and a global min.
- On an **unbounded** domain a global optimum may fail to exist.

Closed-interval recipe. To optimise f on $[a, b]$:

1. Find the stationary points in the **open** interval (a, b).
2. Evaluate f at each of them.
3. Evaluate f at both endpoints, $f(a)$ and $f(b)$.
4. The **largest** value is the global max; the **smallest** is the global min.

> ⚠ Don't Forget the Endpoints
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> The global optimum often sits at a **boundary**. In business, quantities live in $[0, \text{capacity}]$: if the unconstrained optimum lies past the capacity and profit is still rising at the edge, the **binding limit itself is the optimum**.

Profit: Marginal Revenue = Marginal Cost

Profit is revenue minus cost, $\pi(q) = R(q) - C(q)$. The FOC $\pi'(q) = 0$ becomes:

$$\pi'(q) = R'(q) - C'(q) = 0 \quad \Leftrightarrow \quad R'(q) = C'(q)$$

- **In words:** produce while the next unit earns more than it costs; stop where marginal revenue meets marginal cost.
- Confirm a maximum with the SOC: $\pi''(q) < 0$ (the profit curve is concave).
- Fixed costs shift profit but **vanish** under differentiation: they never move the optimal quantity.

Economic Order Quantity (EOQ)

Balance ordering cost against holding cost. With order size q, the annual total is

$$T(q) = \frac{D}{q}K + \frac{q}{2}h \qquad \Rightarrow \qquad \boxed{q^* = \sqrt{\frac{2DK}{h}}}$$

Symbol	Meaning
D	annual demand
K	fixed cost per order
h	holding cost per unit per year
q	order size (the decision)

- The SOC $T''(q) = \frac{2DK}{q^3} > 0$ confirms q^* is a **minimum**.
- At q^* the ordering cost and holding cost are **equal**: that balance is the whole point.
- q^* **rises** with demand D and order cost K, and **falls** as holding cost h increases. Since $q^* \propto \sqrt{D}$, quadrupling demand only doubles the order size.

Optimisation Checklist

> Before You Answer
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> - Did you set $f'(x) = 0$ and find **all** stationary points?
> - Did you classify each with f'' (or the first-derivative test if $f'' = 0$)?
> - On a closed interval, did you also test **both endpoints**?
> - For a business problem: is the answer inside the feasible range $[0, \text{capacity}]$?
> - Did you state the optimum *value*, not just the optimal x or q?