Cheatsheet 04 - Differentiation

Mathematics for Master Students

The Derivative

The **derivative** of f at a is the instantaneous rate of change, the limit of the difference quotient:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- **Geometric reading:** the slope of the **tangent line** at $(a, f(a))$.
- **Business reading:** how fast the output reacts to a tiny nudge at a.
- Sign of f' is a trend report: $f' > 0$ increasing, $f' < 0$ decreasing.
- No derivative at **kinks** (e.g. $|x|$ at 0) or **jumps**: differentiable means a smooth graph.

Notation (all the same thing): $f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x)$.

Differentiation Rules

Building blocks. Memorise these:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	e^x	e^x
x^n	nx^{n-1}	a^x	$a^x \ln a$
$\sqrt{x} = x^{1/2}$	$\frac{1}{2\sqrt{x}}$	$\ln x$	$\frac{1}{x}$

Combination rules. Decompose, then assemble:

Rule	Formula
Constant multiple	$(cf)' = cf'$
Sum	$(f+g)' = f' + g'$
Product	$(fg)' = f'g + fg'$
Quotient	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
Chain	$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

The power rule takes every exponent. Rewrite negatives and roots as powers first:

- $\frac{1}{x} = x^{-1} \Rightarrow \left(\frac{1}{x}\right)' = -x^{-2} = -\frac{1}{x^2}$
- $\sqrt{x} = x^{1/2} \Rightarrow \left(\sqrt{x}\right)' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

Chain rule with e and $\ln$: $\left(e^{kx}\right)' = k\, e^{kx}$, and $(\ln(g(x)))' = \frac{g'(x)}{g(x)}$.

> Name the Structure First
>
> Before differentiating, ask: sum, product, quotient or chain? The right rule follows automatically. The chain rule's inner derivative is not optional: $\left((3x+1)^5\right)' = 5(3x+1)^4 \cdot 3$, never just $5(3x+1)^4$.

Tangent Line

The tangent to f at $x = a$ has slope $f'(a)$ and passes through $(a, f(a))$:

$$y = f(a) + f'(a)(x - a)$$

A **horizontal** tangent ($f'(a) = 0$) marks a peak, a valley, or a flat spot, the starting point for optimisation.

Second Derivative & Curvature

Differentiate twice: $f''(x) = (f')'(x) = \frac{d^2 f}{dx^2}$. While f' reports how fast f changes, f'' reports how fast the **slope** changes.

Condition on an interval	Shape	Chord vs. graph
$f''(x) \geq 0$	**Convex** (bends up)	chord above

Condition on an interval	Shape	Chord vs. graph
$f''(x) \leq 0$	**Concave** (bends down)	chord below

- $f(x) = x^2$: $f'' = 2 > 0$, convex everywhere.
- $f(x) = \ln x$: $f'' = -\frac{1}{x^2} < 0$, concave: diminishing returns.
- Where f'' changes sign, the bend switches: an **inflection point**.

Marginal Analysis

Marginal cost is $C'(q)$, the instantaneous rate at which cost grows at output q:

- Practical reading: approximately the **cost of one more unit** (the $(q + 1)$-th).
- Units: **€ per unit** (a *rate*, not a total). $C'(50) = 60$ means the 51st unit costs about €60, not that 50 units cost €60.
- **Marginal revenue** $R'(q)$: approximately the revenue from one more unit sold. Producing more pays off while $R'(q) > C'(q)$.

Elasticity of Demand

For a demand function $D(p)$, elasticity converts a unit rate into a **percentage** reaction:

$$\varepsilon = \frac{p}{D(p)} \cdot D'(p)$$

A 1% price increase changes demand by about ε percent. Demand falls in price, so $\varepsilon < 0$.

$\lvert\varepsilon\rvert$	Name	Price increase → revenue
> 1	**elastic** (overreacts)	revenue **falls**
< 1	**inelastic** (barely reacts)	revenue **rises**
$= 1$	unit elastic	revenue at its **peak**

Common Pitfalls

⚠ Watch Out

- $(f \cdot g)' \neq f' \cdot g'$: use the product rule $f'g + fg'$.
- Chain rule: never forget the **inner derivative**. Multiply by the derivative of the inside.
- Rewrite $\frac{1}{x}$ and $\sqrt{x}$ as powers *before* applying the power rule.
- $C'(q)$ is a rate in **€ per unit**, not a total cost in €.
- Elasticity is **negative**; classify by $\lvert\varepsilon\rvert$ against 1.